Lecture 3

Game Theory
Extensive Form Games:
Backward Induction/Subgame Perfect Equilibrium

(JR 7.3.5-7.3.6)

MSc Economics

EC7001 - Microeconomics

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Games of Perfect Information

Introduction

- Games of perfect information: all players are perfectly informed of all previous actions taken whenever it is their turn to move
- Example 1: Incumbent-Entrant game
  - 2 firms competing in a single industry; one currently producing, one not
  - The entrant must decide whether to enter the industry or to stay out
  - If the entrant stays out, the status quo prevails $(0, 2)$ and the game ends
  - If the entrant enters, the incumbent must decide whether to fight by driving the price down $(-1, -1)$ or to acquiesce $(1, 1)$
    (check game tree)
- Example 2 (check game tree)

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Backward Induction Strategies

- We say that y strictly follows x if $y = (x, a_1, \ldots, a_k)$ for some $a_1, \ldots, a_k \in A$ and that y immediately follows x if $k = 1$; we say that y weakly follows x if $y = x$ or y strictly follows x

Definition (**Backward Induction Strategies**): The joint (pure) strategy s is a backward induction strategy for the finite extensive form game Γ of perfect information if it is derived as follows:

1. Call a node x penultimate in Γ if all nodes immediately following it are end nodes
2. For every penultimate node x, let $s_{\iota(x)}(x)$ be an action leading to an end node that maximizes player $\iota(x)$'s payoff from among the actions at x
3. Let u_x denote the resulting payoff vector
4. Remove the nodes and actions strictly following each penultimate node x in Γ and assign the payoff u_x to x, which then becomes an end node in Γ
5. Repeat until an action has been assigned to every decision node

(finite)

6. This then yields a backward induction joint pure strategy s

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Backward Induction and Nash Equilibrium

Theorem (**Backward Induction and NE**): If s is a backward induction strategy for the perfect information finite extensive form game Γ, then s is a Nash Equilibrium of Γ

Proof: check at home; by contradiction: suppose s is not a NE, then it is not a backward induction strategy

Corollary (**Existence of Pure Strategy NE**): (Because the backward induction algorithm always terminates in finite games with perfect information) every finite extensive form game of perfect information possesses a pure strategy NE

- Although every backward induction strategy is a NE, not every NE is a backward induction strategy (back to Incumbent-entrant game)

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Introduction

- The backward induction technique does not however extend immediately to games of imperfect information

  (check coordination game with an option; check subgame)

- In games of imperfect information (contrary to those of perfect information) we must to some extent simultaneously determine optimal play at points both earlier and later in the game

  (check a set of subgame perfect equilibrium strategies for the coordination game)

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Subgames

Definition (**Subgames**): A node x is said to define a subgame of an extensive form game if whenever y is a decision node weakly following x and z is in the information set containing y, then z also weakly follows x

(In other words: a node x defines a subgame if every player on every turn knows whether x has been reached)

- Note that the definition encompasses the case where $y = x$, i.e. where x forms a singleton information set $\Psi(x) = \{x\}$

(check examples)

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Pure Strategy Subgame Perfect Equilibrium

Definition (**Pure Strategy Subgame Perfect Equilibrium**): A subgame defined by a node x in the extensive form game Γ is denoted Γ_x; a joint pure strategy s is a pure strategy subgame perfect equilibrium of the extensive form game Γ if s induces a NE in every subgame Γ_x (i.e. for all nodes x that define a subgame in Γ)

- Note that, because for any extensive form game Γ the game itself is a subgame, a pure strategy subgame perfect equilibrium of Γ is also a pure strategy NE of Γ
- Pure strategies SPE is a refinement of the NE concept (i.e. not every NE is a pure strategies SPE)
  (check example)

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Subgame Perfect Equilibrium vs. Backward Induction

Theorem (**SPE generalizes Backward Induction**): For every finite extensive form game of perfect information, the set of backward induction strategies coincides with the set of pure strategy SPE

Sketch of Proof:

1. Every backward induction strategy is subgame perfect: s is a backward induction strategy; as the game has perfect information, the strategy induced by s at any subgame is still a backward induction strategy; we just need to apply last theorem ('any BIS is a NE') to all subgames
2. Every pure strategy subgame perfect equilibrium is a backward induction strategy: s is subgame perfect; we verify that s can be derived through the backward induction algorithm, starting from penultimate nodes and using payoff-maximizing property of NE

- Some games may not have pure strategy SPE; just like in strategic form games (check example); to guarantee existence, we need randomization

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Mixed Strategies vs. Behavioral Strategies

- There are two ways one might randomize behavior in an extensive form game:

  1. Randomize once and for all at the beginning of the game (direct analogue of strategic form games) - mixed strategy

  2. Employ randomization whenever it is one's turn to move (over current - at that turn - set of available actions) - behavioral strategy

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Mixed Strategies vs. Behavioral Strategies (cont'ed)

Definition (**Mixed Strategies**): A mixed strategy m_i for player i is a probability distribution over player i's set of pure strategies S_i; for each pure strategy $s_i \in S_i$, $m_i(s_i)$ denotes the probability that the pure strategy s_i is chosen; $m_i(s_i) \in [0,1]$ and $\sum_{s_i \in S_i} m_i(s_i) = 1$

Definition (**Behavioral Strategies**): A behavioral strategy b_i provides for each of player i's informational sets a probability distribution over the actions available there; $b_i(a,I) \in [0,1]$ and $\sum_{a \in A(I)} b_i(a,I) = 1$, for every information set I belonging to player i, where $A(I)$ is the set of actions available at information set I

- Mixed and behavioral strategies are strategically equivalent under perfect recall

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Perfect Recall

Definition (**Perfect recall**): An extensive form game has perfect recall if whenever two nodes x and $y = (x, a, a_1, \ldots, a_k)$ belong to a single player, then every node in the same information set as y is of the form $w = (z, a, a'_1, \ldots, a'_k)$ for some node z in the same information set as x

(In other words: perfect recall says that each player always remembers what she knew in the past about the history of play; any two histories y and w that a player's information set does not allow that player to distinguish can differ only in the actions taken by other players; no player ever forgets an action that she has taken in the past

(check example)

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Subgame Perfect Equilibrium

Definition (**Subgame Perfect Equilibrium**): A joint behavioral strategy (wlog) b is a subgame perfect equilibrium of the finite extensive form game Γ if it induces a NE in every subgame of Γ

- The equivalence between behavioral and mixed strategies enables us to:
  - use our earlier result concerning mixed strategies in strategic form games: a behavioral strategy is a NE if no player has a pure strategy giving a higher payoff given the behavioral strategies of the others
  - every SPE is a NE

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Existence of Subgame Perfect Equilibrium

Theorem (**Existence of Subgame Perfect Equilibrium**): Every finite extensive form game with perfect recall possesses a SPE

Sketch of Proof: The idea of the proof is to use the backward induction algorithm (by constructing a behavioral strategy from the end of the game to the beginning, by using existence of NE in mixed strategies along the way, and by applying equivalence between mixed and behavioral strategies)