Lecture 4

Game Theory
Extensive Form Games: Sequential Equilibrium

(JR 7.3.7)

MSc Economics
EC7001 - Microeconomics

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Sequential Equilibrium

Introduction

- Not all subgame perfect equilibria are sensible
  (check example)

- We need to introduce beliefs for the players over the nodes within their information sets for the purposes of refining the set of subgame perfect equilibria (like we did with backward induction over NE)

Sequential Equilibrium
Beliefs

- Given an extensive form game Γ and a decision node x, let $p(x)$ denote the probability that player $\iota(x)$ assigns to the history x conditional on his information set $\Psi(x)$ having been reached
- We must have $\sum_{y \in \Psi(x)} p(y) = 1$ for every decision node x
- The function $p(.)$ is called a *system of beliefs* because it embodies the beliefs of all players at each of their information sets regarding the history of play up to that point in the game

  (in a tree diagram, we represent the system of beliefs, p, by placing the probability assigned to each node in square brackets)
- We denote an *assessment* as a system of beliefs/behavioral strategy pair (p, b); given such an ordered pair, the beliefs p are interpreted as those that are held by the players given that the behavioral strategy b is being played; which assessments are sensible?

  (check example)

Sequential Equilibrium

Beliefs - Principles

- For an assessment (p, b) to be sensible, the system of beliefs p ought to be derived from the given joint behavioral strategy b using Bayes' rule whenever possible
- Letting $P(x|b)$ denote the probability that node x is reached given the behavioral strategy b, Bayes' rule states that for every information set I, and every $x \in I$,

$$p(x) = \frac{P(x|b)}{\sum_{y \in I} P(y|b)},$$

whenever the denominator is positive, that is, whenever the information set is reached with positive probability according to b

I: **Beliefs must be derived from strategies using Bayes' rule when possible**

II: **Beliefs must reflect that players choose their strategies independently**

III: **Players with identical information have identical beliefs**

(check examples)

Sequential Equilibrium

Beliefs - Consistency

- An assessment satisfies Bayes' rule, independence and common beliefs if and only if it is consistent (proof ommited, see Kohlberg and Reny, 1997)
- More terminology: a behavioral strategy in a finite extensive form game is called *completely mixed* if it assigns strictly positive probability to every action at every information set (for these strategies, every information set is reached with positive probability, meaning Bayes' rule alone determines beliefs)

Definition (**Consistent Assessments**): An assessment (p, b) for a finite extensive form game Γ is consistent if there is a sequence of completely mixed behavioral strategies, b^n, converging to b, such that the associated sequence of Bayes' rule induced systems, p^n, converges to p

(check examples)

Sequential Equilibrium

Sequential Rationality

- Now that we have endowed each player with beliefs about the history of play whenever it is that player's turn to move, it is straightforward to require that the choices made at each information set be optimal there (just like for backward induction under perfect information)
- Fix a finite extensive form game; consider an assessment (p, b), and an information set I, belonging to player i
- We now calculate i's payoff according to the assessment (p, b) given that his information set I has been reached; for each node x in I, let $u_i(b|x)$ denote i's payoff for playing b; of course, player i does not know which node within I has been reached; but the system of beliefs p describes the probabilities that i assigns to each node in I
- Player i's payoff accoring to (p, b) given that I has been reached is simply the expected value of the numbers $u_i(b|x)$ according to the system of beliefs p, i.e. $v_i(p, b|I) = \sum_{x \in I} p(x) u_i(b|x)$

  (check example)

Sequential Equilibrium

Sequential Rationality - Definition

Definition (**Sequential Rationality**): An assessment (p, b) for a finite extensive form game is sequentially rational if for every player i, every information set I belonging to player i, and every behavioral strategy b_i' of player i,

$$v_i(p, b|I) \geq v_i(p, (b_i', b_{-i}|I);$$

we call a joint behavioral strategy b, sequentially rational if for some system of beliefs p, the assessment (p, b) is sequentially rational

(in other words: an assessment is sequentially rational if no player, at any point in the game, ever has an incentive to change his strategy; any point in the game meaning all information sets, including those with zero prob as given by b)

(remember first game example in the lecture: SPE depicted was not sensible because there would be a deviation in case information set by player 2 was reached; but this information set had probability zero under the SPE) (not all SPE (NE) are sequentially rational; opposite? - check example)

Sequential Equilibrium

Definition

Definition (**Sequential Equilibrium**): An assessment for a finite extensive form game is a sequential equilibrium if it is both consistent and sequentially rational

Theorem (**Existence of Sequential Equilibrium**): Every finite extensive form game with perfect recall possesses at least one sequential equilibrium; moreover, if an assessment (p, b) is a sequential equilibrium, then the behavioral strategy b is a subgame perfect equilibrium